Some Approximation Properties of the Generalized Baskakov operators

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Abstract

The present paper deals with a generalization of the Baskakov operators. Some direct theorems, asymptotic formula and A-statistical convergence are established. Our results are based on a ρ function. These results include the preservation properties of the classical Baskakov operators.

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1. Introduction

In [2], Baskakov discussed the following positive linear operators on the unbounded interval $[0, \infty)$,

$$V_n(f; x) = \sum_{k=0}^{\infty} v_{k,n}(x) f\left(\frac{k}{n}\right), \quad x \in [0, \infty), n \in \mathbf{N}, \tag{1.1}$$

where f is an appropriate function defined on the unbounded interval $[0, \infty)$, for which the above series is convergent and $v_{n,k}(x) = \binom{n+k-1}{k} \dfrac{x^k}{(1+x)^{n+k}}$.

In 2011, Cárdenas-Morales *et al.* [3] introduced a generalized Bernstein operators by fixing e_0 and e_1, given by

$$L_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^{2k} (1-x^2)^{n-k} f\left(\sqrt{\frac{k}{n}}\right), \quad x \in [0, 1], n \in \mathbf{N}, \tag{1.2}$$

where $f \in C[0, 1]$. This is a special case of the operators $B_n^\tau f = B_n \left(f \circ \tau^{-1}\right) \circ \tau$, for $\tau = e_2$, where B_n is the classical Bernstein operators.

Consider a real valued function ρ on $[0, \infty)$ satisfied following two conditions:

1. ρ is a continuously differentiable function on $[0, \infty)$,

2. $\rho(0) = 0$ and $\displaystyle\inf_{x \in [0, \infty)} \rho'(x) \geq 1$.

Throughout the manuscript, we denote the above two conditions as c_1 and c_2. Recently, In [1] the following generalization of Szász–Mirakyan operators are constructed,

$$S_n^\rho(f;x) = \exp\left(-n\rho(x)\right)\sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\frac{(n\rho(x))^k}{k!}, \quad x\in[0,1], n\in\mathbf{N}. \tag{1.3}$$

Notice that if $\rho = e_1$ then the operators (1.3) reduces to the well known Szász–Mirakyan operators. Aral *et al.* [1], gave quantitative type theorems in order to obtain the degree of weighted convergence with the help of a weighted modulus of continuity constructed using the function ρ of the operators (1.3). Very recently, some researchers have discussed shape preserving properties of the generalized Bernstein, Baskakov and Szász-Mirakjan operators in [17–20].

2. Construction of the Operators

This motivated us to generalize the Baskakov operators (1.1) as

$$\begin{aligned}
V_n^\rho(f;x) &= \sum_{k=0}^{\infty}\left(f\circ\rho^{-1}\right)\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{(\rho(x))^k}{(1+\rho(x))^{n+k}} \\
&= \left(V_n^\rho\left(\left(f\circ\rho^{-1}\right)\circ\rho\right)\right)(x) \\
&= \sum_{k=0}^{\infty}f\left(\rho^{-1}\left(\frac{k}{n}\right)\right)v_{\rho,n,k}(x),
\end{aligned} \tag{2.1}$$

where $n\in\mathbf{N}$, $x\in[0,\infty)$ and ρ is a function defined as in conditions c_1 and c_2.
Observe that, $V_n^\rho(f;x) = V_n(f;x)$ if $\rho = e_1$. In fact, direct calculation gives that

$$V_n^\rho(e_0;x) = 1; \tag{2.2}$$

$$V_n^\rho(\rho;x) = \rho(x); \tag{2.3}$$

$$V_n^\rho(\rho^2;x) = \rho^2(x) + \frac{\rho^2(x)+\rho(x)}{n}. \tag{2.4}$$

In this manuscript, we are dealing with approximation properties the operators (2.1). In the next section, we establish some direct results using generalized modulus of continuity. The Voronovskaya Asymptotic formula and A-Statistical convergence of the operators V_n^ρ are discuss in Section 4 and 5.

3. Direct Theorems

Consider $\phi^2(x) = 1 + \rho^2(x)$. Notice that, $\lim_{x\to\infty}\rho(x) = \infty$ because the condition c_2. Denote $B_\phi\left([0,\infty)\right)$ as set of all real valued function on $[0,\infty)$ such that $|f(x)| \leq M_f\phi(x)$, for all $x\in[0,\infty)$, where M_f is a constant depending on f. Observe that, $B_\phi\left([0,\infty)\right)$ is norm linear space with the norm $\|f\|_\phi = \sup\left\{\frac{|f(x)|}{\phi(x)} : x\in[0,\infty)\right\}$. Also, $C_\phi\left([0,\infty)\right)$ as the set all continuous function in $B_\phi\left([0,\infty)\right)$ and

$$C_\phi^k\left([0,\infty)\right) = \{f\in C_\phi\left([0,\infty)\right) : \lim_{x\to\infty}\frac{f(x)}{\phi(x)} = k_f, k_f \text{ is constant depaneding on } f\}.$$

Let $U_\phi([0, \infty))$ be the space of functions $f \in C_\phi([0, \infty))$, such that $\dfrac{f(x)}{\phi(x)}$ is uniformly continuous. Also, $C_\phi^k([0, \infty)) \subset U_\phi([0, \infty)) \subset C_\phi([0, \infty)) \subset B_\phi([0, \infty))$.

Lemma 1. *[11] The positive linear operators $L_n : C_\phi([0, \infty)) \to B_\phi([0, \infty))$ for all $n \geq 1$ if and only if the inequality*

$$|L_n(\phi; x)| \leq K_n \phi(x), \quad x \in [0, \infty), \ n \geq 1,$$

holds, where K_n is a positive constant.

Theorem 1. *[11] Let the sequence of linear positive operators $(L_n)_{n \geq 1}$, $L_n : C_\phi([0, \infty)) \to B_\phi([0, \infty))$ satisfy the three conditions*

$$\lim_{n \to \infty} \|L_n \rho^i - \rho^i\| = 0, \quad i = 0, 1, 2.$$

Then

$$\lim_{n \to \infty} \|L_n f - f\| = 0,$$

for any $f \in C_\phi^k([0, \infty))$.

By (2.2), (2.4) and Lemma 1, V_n^ρ is linear positive operators from $C_\phi([0, \infty))$ to $B_\phi([0, \infty))$.

Theorem 2. *For each function $f \in C_\phi^k([0, \infty))$, we have*

$$\lim_{n \to \infty} \|V_n^\rho(f; \cdot) - f\|_\phi = 0. \tag{3.1}$$

Proof: From (2.2) and (2.3), we write

$$\|V_n^\rho(1; \cdot) - 1\|_\phi = 0 \text{ and } \|V_n^\rho(\rho; \cdot) - \rho\|_\phi = 0.$$

Also,

$$\|V_n^\rho(\rho^2; \cdot) - \rho^2\|_\phi = \sup_{x \in [0, \infty)} \frac{\rho^2(x) + \rho(x)}{n(1 + \rho^2(x))} \leq \frac{2}{n}. \tag{3.2}$$

Therefore, we have

$$\|V_n^\rho(\rho^i; \cdot) - \rho^i\|_\phi \to 0 \text{ as } n \to \infty, \text{ for } i = 0, 1, 2.$$

Hence by Theorem 1, the equation (3.1) is also true.

In [13] the following weighted modulus of continuity is defined

$$\omega_\rho(f; \delta) := \omega(f; \delta)_{[0, \infty)} = \sup_{\substack{x, t \in [0, \infty) \\ |\rho(x) - \rho(t)| \leq \delta}} \frac{|f(t) - f(x)|}{|\phi(t) - \phi(x)|} \tag{3.3}$$

for each $f \in C_\phi([0, \infty))$ and for every $\delta > 0$.

We call the function $\omega_\rho(f; \delta)$ weighted modulus of continuity. We observe that $\omega_\rho(f; 0) = 0$ for every $f \in C_\phi([0, \infty))$ and that the function $\omega_\rho(f; \delta)$ is nonnegative and nondecreasing with respect to δ for $f \in C_\phi([0, \infty))$. Here, we

consider the spaces $C_\phi^k([0, \infty))$, $U_\phi([0, \infty))$, $C_\phi([0, \infty))$ and $B_\phi([0, \infty))$ having the conditions c_1 and c_2. Under these conditions, $|x - t| \leq |\rho(x) - \rho(t)|$, for every $x, t \in [0, \infty)$ is true.

Lemma 2. *[13]* $\lim\limits_{\delta \to 0} \omega_\rho(f; \delta) = 0$, *for every* $f \in U_\phi([0, \infty))$.

Theorem 3. *[13] Let* $L_n : C_\phi([0, \infty)) \to B_\phi([0, \infty))$ *be a sequence of positive linear operators with*

$$\|L_n(\rho^0) - \rho^0\|_{\phi^0} = a_n, \tag{3.4}$$

$$\|L_n(\rho) - \rho\|_{\phi^{\frac{1}{2}}} = b_n, \tag{3.5}$$

$$\|L_n(\rho^2) - \rho^2\|_{\phi} = c_n, \tag{3.6}$$

$$\|L_n(\rho^3) - \rho^3\|_{\phi^{\frac{3}{2}}} = d_n, \tag{3.7}$$

where a_n, b_n, c_n *and* d_n *tends to zero as* $n \to \infty$. *Then*

$$\|L_n(f) - f\|_{\phi^{\frac{3}{2}}} = (7 + 4a_n + 2c_n)\omega_\rho(f; \delta_n) + a_n\|f\|_\phi,$$

for all $f \in C_\phi([0, \infty))$, *where*

$$\delta_n = 2\sqrt{(a_n + 2b_n + c_n)(1 + a_n)} + a_n + 3b_n + 3c_n + d_n.$$

Theorem 4. *For all* $f \in C_\phi([0, \infty))$, *we have*

$$\|V_n^\rho(f; \cdot) - f\|_{\phi^{\frac{3}{2}}} \leq \left(7 + \frac{4}{n}\right)\omega_\rho\left(f, \frac{2\sqrt{2}}{\sqrt{n}} + \frac{16}{n}\right).$$

Proof: Notice that

$$\|V_n^\rho(\rho^0; \cdot) - \rho^0\|_{\phi^0} = 0 = a_n \tag{3.8}$$

and

$$\|V_n^\rho(\rho; \cdot) - \rho\|_{\phi^{\frac{1}{2}}} = 0 = b_n. \tag{3.9}$$

Form equation (3.2), we have

$$c_n = \|V_n^\rho(\rho^2; \cdot) - \rho^2\|_\phi \leq \frac{2}{n}. \tag{3.10}$$

Now,

$$V_n^\rho(\rho^3; x) = \frac{\rho(x)}{n^2} + \frac{3(1 + n)\rho(x)^2}{n^2} + \frac{(2 + 3n + n^2)\rho(x)^3}{n^2}. \tag{3.11}$$

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We can write

$$d_n = \|V_n^\rho(\rho^3;\cdot) - \rho^3\|_{\phi^{\frac{3}{2}}} = \sup_{x\in[0,\infty)} \left\{ \frac{\rho(x)}{n^2\left(1+\rho^2(x)\right)^{\frac{3}{2}}} \right.$$

$$\left. + \frac{3(1+n)\rho(x)^2}{n^2\left(1+\rho^2(x)\right)^{\frac{3}{2}}} + \frac{(2+3n)\rho(x)^3}{n^2\left(1+\rho^2(x)\right)^{\frac{3}{2}}} \right\}$$

$$\leq \frac{1}{n} + \frac{4}{n} + \frac{5}{n} = \frac{10}{n}.$$

Observe that, the condition (3.4)-(3.7) are satisfied, therefore by theorem 3, we have

$$\|V_n^\rho(f;\cdot) - f\|_{\phi^{\frac{3}{2}}} \leq \left(7 + \frac{4}{n}\right)\omega_\rho\left(f, \frac{2\sqrt{2}}{\sqrt{n}} + \frac{16}{n}\right).$$

This completes the proof of Theorem 4.

Theorem 5. *For all* $f \in U_\phi^k\left([0,\infty)\right)$, *we have* $\lim_{n\to\infty} \|V_n^\rho(f;\cdot) - f\|_{\phi^{\frac{3}{2}}} = 0$.

The proof follows from the Theorem 4 and Lemma 2.

4. Voronovskaya Asymptotic formula

Now we give the following Voronovskaya type theorem. We use the technique developed in [1, 3].

Theorem 6. *Let* $f \in C[0,\infty)$, $x \in [0,\infty)$ *and suppose that the first and second derivatives of* $f \circ \rho^{-1}$ *exist at* $\rho(x)$. *If the second derivative of* $f \circ \rho^{-1}$ *is bounded on* $[0,\infty)$ *then we have*

$$\lim_{n\to\infty} n(V_n^\rho(f;x) - f(x)) = \frac{1}{2}\rho(x)(1+\rho(x))\left(f\circ\rho^{-1}\right)''(\rho(x)). \tag{4.1}$$

Proof: By the Taylor expansion of $f \circ \rho^{-1}$ at the point $\rho(x) \in [0,\infty)$, there exists ξ lying between x and t such that

$$f(t) = \left(f\circ\rho^{-1}\right)(\rho(t)) = \left(f\circ\rho^{-1}\right)(\rho(x)) + \left(f\circ\rho^{-1}\right)'(\rho(x))(\rho(t) - \rho(x))$$

$$+\frac{1}{2}\left(f\circ\rho^{-1}\right)''(\rho(x))(\rho(t) - \rho(x))^2 + \lambda_x(t)(\rho(t) - \rho(x))^2, \tag{4.2}$$

where

$$\lambda_x(t) = \frac{\left(f\circ\rho^{-1}\right)'(\rho(\xi)) - \left(f\circ\rho^{-1}\right)''(\rho(x))}{2}. \tag{4.3}$$

Note that, the assumptions on f together with definition (4.3) ensure that $|\lambda_x(t)| \leq M$ for all t and converges to zero as $t \to x$.

Applying the operators (2.1) to the above equation (4.2) equality, we get

$$V_n^\rho(f;x) - f(x) = \left(f\circ\rho^{-1}\right)'(\rho(x)) V_n^\rho\left((\rho(t) - \rho(x));x\right)$$

$$+\frac{1}{2}\left(f\circ\rho^{-1}\right)''(\rho(x)) V_n^\rho\left((\rho(t) - \rho(x))^2;x\right)$$

$$+V_n^\rho\left(\lambda_x(t)(\rho(t) - \rho(x))^2;x\right), \tag{4.4}$$

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by equations (2.2), (2.3) and (2.4), we have

$$\lim_{n\to\infty} nV_n^\rho\left((\rho(t)-\rho(x))\,;x\right)=0;$$

$$\lim_{n\to\infty} nV_n^\rho\left((\rho(t)-\rho(x))^2\,;x\right)=\rho(x)(1+\rho(x)).$$

Therefore,

$$\lim_{n\to\infty} n\left(V_n^\rho(f;x)-f(x)\right) = \frac{1}{2}\rho(x)(1+\rho(x))\left(f\circ\rho^{-1}\right)''(\rho(x))$$
$$+ \lim_{n\to\infty} n\left(V_n^\rho\left(\lambda_x(t)\,(\rho(t)-\rho(x))^2\,;x\right)\right). \tag{4.5}$$

Now we estimate the last term on the right hand side of the above equality. Let $\epsilon>0$ and choose $\delta>0$ such that $|\lambda_x(t)|<\epsilon$ for $|t-x|<\delta$. Also it is easily seen that by condition c_2, $|\rho(t)-\rho(x)|=\rho(\eta)|t-x|\geq|t-x|$. Therefore, if $|\rho(t)-\rho(x)|<\delta$, then $|\lambda_x(t)(\rho(t)-\rho(x))^2|<\epsilon\left(\rho(t)-\rho(x)\right)^2$, while if $|\rho(t)-\rho(x)|\geq\delta$, then since $|\lambda_x(t)|\leq M$ we have $|\lambda_x(t)\left(\rho(t)-\rho(x)\right)^2|\leq\frac{M}{\delta^2}\left(\rho(t)-\rho(x)\right)^4$. So we can write

$$V_n^\rho\left(\lambda_x(t)\,(\rho(t)-\rho(x))^2\,;x\right) < \epsilon\left(V_n^\rho\left((\rho(t)-\rho(x))^2\,;x\right)\right)$$
$$+\frac{M}{\delta^2}\left(V_n^\rho\left((\rho(t)-\rho(x))^4\,;x\right)\right). \tag{4.6}$$

Direct calculations show that

$$V_n^\rho\left((\rho(t)-\rho(x))^4\,;x\right)=O\left(\frac{1}{n^2}\right).$$

Hence

$$\lim_{n\to\infty} nV_n^\rho\left(\lambda_x(t)\,(\rho(t)-\rho(x))^2\,;x\right)=0,$$

which completes the proof of the Theorem 8.

5. A-Statistical Convergence

Now, we introduces some notation and the basic definitions, which used in this section. Let $A=(a_{ij})$ be an infinite summability matrix. For given sequence $x=(x_n)$, the $A-$transform to x, denoted by $Ax=\left((Ax)_j\right)$, is given by $(Ax)_j=\sum_{n=1}^{\infty} a_{jn}x_n$, provided the series converges for each j. We say that A is regular, if $\lim_j(Ax)_j=L$ whenever $\lim_j x_j=L$ [12].

Now, we assume that A is a nonnegative regular summability matrix and K is a subset of $\mathbf{N}$, the set of all natural numbers. The A-density of K is defined by $\delta_A(K)=\lim_j\sum_{n=1}^{\infty} a_{jn}\chi_K(n)$ provided the limit exists, where χ_K is the characteristic function of K. Then the sequence $x=(x_n)$ is said to be A-statistically convergent to the number L if, for every $\epsilon>0$, $\delta_A\{n\in\mathbf{N}:|x_n-L|\geq\epsilon\}=0$; or equivalently $\lim_j\sum_{n:|x_n-L|\geq\epsilon} a_{jn}=0$. We denote this limit by $st_A-lim\,x=L$ [4, 5, 8, 16].

For the case in which $A=C_1$, the Cesàro matrix, A-statistical convergence reduces to statistical convergence [7, 9, 10]. Also, taking $A=I$, the identity matrix, A-statistical convergence coincides with the ordinary convergence.

We also note that if $A = (a_{jn})$ is a nonnegative regular summability matrix for which $\lim_{j} \max_{n}\{a_{jn}\} = 0$, then A-statistical convergence is stronger than convergence [15]. A sequence $x = (x_n)$ is said to be A-statistically bounded provided that there exists a positive number M such that $\delta_A\{n \in \mathbf{N} : |x_n| \leq M\} = 1$. Recall that $x = (x_n)$ is A-statistically convergent to L if and only if there exists a subsequence $x_{n(k)}$ of x such that $\delta_A\{n(k) : k \in \mathbf{N}\} = 1$ and $\lim_{k} x_{n(k)} = L$ (see [15, 16]). Note that, the concept of A-statistical convergence is also given in normed spaces [14].

In this section, we denote $B_\phi([0,\infty))$ by B_ϕ and $C_\phi([0,\infty))$ by C_ϕ. Assume $\phi_1(x)$ and $\phi_2(x)$ be weight functions satisfying $\lim_{|x|\to\infty} \dfrac{\phi_1(x)}{\phi_2(x)} = 0$. If T is a positive operators such that $T : C_{\phi_1} \to B_{\phi_2}$, then the operators norm $\|T\|_{C_{\phi_1}\to B_{\phi_2}}$ is given by $\|T\|_{C_{\phi_1}\to B_{\phi_2}} := \sup_{\|f\|_{\phi_1}=1} \|Tf\|_{\phi_2}$.

Theorem 7. *[6, Thorem 6] Let $A = (a_{jn})$ be a non-negative regular summability matrix, let $\{T_n\}$ be a sequence of positive linear operators from C_{ϕ_1} into B_{ϕ_2} and assume that $\phi_1(x)$ and $\phi_2(x)$ be weight functions satisfying $\lim_{|x|\to\infty} \dfrac{\phi_1(x)}{\phi_2(x)} = 0$. Then*

$$st_A - \lim_{n} \|T_n f - f\|_{\phi_2} = 0, \ \ for \ all \ f \in C_{\phi_1} \tag{5.1}$$

if and only if

$$st_A - \lim_{n} \|T_n \rho^v - \rho^v\|_{\phi_1} = 0, \quad v = 0, 1, 2. \tag{5.2}$$

With the help of Theorem 7 we write the following Korovkin type theorem.

Theorem 8. *Let $A = (a_{jn})$ be a non-negative regular summability matrix, let $\{V_n\}$ be a sequence of positive linear operators from C_{ϕ_1} into B_{ϕ_2} as defined in (2.1) and assume that $\phi_1(x)$ and $\phi_2(x)$ be weight functions satisfying $\lim_{|x|\to\infty} \dfrac{\phi_1(x)}{\phi_2(x)} = 0$. Then*

$$st_A - \lim_{n} \|V_n(f, \cdot) - f\|_{\phi_2} = 0, \ \ for \ all \ f \in C_{\phi_1}. \tag{5.3}$$

Proof: By theorem 7 it is sufficient to prove that,

$$st_A - \lim_{n} \|V_n(\rho^v, \cdot) - \rho^v\|_{\phi_1} = 0, \quad v = 0, 1, 2. \tag{5.4}$$

It clear that

$$\|V_n(\rho^0, \cdot) - \rho^0\|_{\phi_1} = 0 \text{ and } \|V_n(\rho, \cdot) - \rho\|_{\phi_1} = 0.$$

Hence, equation (5.4) is true for $v = 0, 1$.

Now, for $v = 2$

$$\|V_n(\rho^2, \cdot) - \rho^2\|_{\phi_1} \leq \frac{2}{n}. \tag{5.5}$$

Due to the equality $st_A - \lim_{n} \dfrac{1}{n} = 0$, the above inequality implies that

$$st_A - \lim_{n} \|V_n(\rho^2, \cdot) - \rho^2\|_{\phi_1} = 0, \tag{5.6}$$

which completes the proof the Theorem 8.

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Conclusions

We constructed sequences of the Baskakov operators which are based on a function ρ. This function not only characterizes the operators but also characterizes the Korovkin set $\{1, \rho, \rho^2\}$ in a weighted function space.

Our results include the following: The rate of convergence of these operators to the identity operator on weighted spaces which are constructed using the function ρ and which are subspaces of the space of continuous functions on $[0, \infty)$. We gave quantitative type theorems in order to obtain the degree of weighted convergence with the help of a weighted modulus of continuity constructed using the function ρ and the study of A-statistical convergence of the sequence.

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